

# Effective field theories for baryons with two- and three-heavy quarks

Antonio Vairo

Received: date / Accepted: date

**Abstract** Baryons made of two or three heavy quarks can be described in the modern language of non-relativistic effective field theories. These, besides allowing a rigorous treatment of the systems, provide new insight in the nature of the three-body interaction in QCD.

**Keywords** QCD · Effective Field Theories · Heavy Quarks · Baryons

## 1 Motivations

Baryons made of two or three heavy quarks offer an interesting alternative to quarkonium for studying the dynamics of non-relativistic systems in QCD and for investigating the transition region from Coulombic to confined bound states.

The modern approach to quarkonium physics consists in taking advantage of the hierarchy of non-relativistic energy scales in the system by constructing a suitable hierarchy of effective field theories (EFTs) [1]. The energy scales are the heavy-quark mass,  $m$ , the typical momentum transfer,  $p \ll m$ , whose inverse sets the typical distance,  $r$ , between the heavy quark and the antiquark, and the typical kinetic energy,  $E \ll p$ , whose inverse sets the typical time scale of the bound state. In the ultimate EFT, obtained after integrating out gluons of energy and momentum of the order of  $m$  and  $p$ , the interaction between heavy quarks is organized as an expansion in powers of  $1/m$  and  $r$ . At zeroth-order in  $r$ , the interaction is entirely encoded in the quark-antiquark potential, which, at zeroth-order in  $1/m$ , reduces to the static potential. Terms proportional to powers of  $1/m$  and  $r$  can be systematically added.

An analogous hierarchy of energy scales characterizes also baryons that contain at least two heavy quarks. Hence, we may describe these systems by means of EFTs analogous to the ones suited for heavy quarkonium [2,3]. In particular, the ultimate

---

“Relativistic Description of Two- and Three-Body Systems in Nuclear Physics”, ECT\*, October 19-23 2009

---

A. Vairo  
Physik-Department, Technische Universität München, James-Frank-Str. 1, 85748 Garching, Germany  
E-mail: antonio.vairo@tum.de

EFT, obtained after integrating out gluons of energy and momentum of the order of the heavy-quark masses and of the typical momentum transfer between heavy quarks, is organized as an expansion in the inverse of the heavy-quark masses and in the distance between the heavy quarks. For equal heavy-quark masses, the structure of the Lagrangian and its power counting are similar to the quarkonium case; the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \sum_f \bar{q}_f i \not{D} q_f + \sum_{i,j} \delta\mathcal{L}^{(i,j)}, \quad (1)$$

where  $\delta\mathcal{L}^{(i,j)}$  are terms containing the heavy quark or antiquark fields, which are proportional to  $1/m^i \times [\text{typical distance between heavy quarks/antiquarks}]^j$ . The fields  $q_f$  are  $n_f$  light-quark fields, assumed to be massless. The heavy quark or antiquark fields in  $\delta\mathcal{L}^{(i,j)}$  are twice those in the bound state, e.g, in the quarkonium case,  $\delta\mathcal{L}^{(i,j)}$  contains two quark and two antiquark fields. For different heavy-quark masses, more scales are involved; in the following, we will not consider such cases.

Although the EFTs for quarkonium and baryons made of two or three heavy quarks are similar in structure, they are characterized by different degrees of freedom. This is best seen when the typical distance between heavy quarks is smaller than the inverse of the hadronic scale  $\Lambda_{\text{QCD}}$ , which is the case that we will discuss in this note. At these distances, gluons may resolve coloured degrees of freedom. These are gluons or light quarks or, in the quarkonium case,  $Q\bar{Q}$  states in a colour singlet or in a colour octet configuration. In the case of baryons made of two heavy quarks  $Q$  and a light quark  $q$ , at distances smaller than  $1/\Lambda_{\text{QCD}}$ , gluons can resolve gluons, light quarks and  $QQ$  pairs in a colour antitriplet or in a colour sextet configuration. It is the binding of the antitriplet with the light quark  $q$  that forms the  $QQq$  baryon. The system very much resembles a heavy-light meson, with the heavy antiquark replaced by a  $QQ$  antitriplet; this fact may be exploited to deduce some properties of the  $QQq$  baryons from the corresponding  $\bar{Q}q$  mesons [4]. In the case of baryons made of three heavy quarks  $Q$ , at distances smaller than  $1/\Lambda_{\text{QCD}}$ , gluons can resolve gluons, light quarks and  $QQQ$  states either in a colour singlet or in two different colour octets or in a colour decuplet configuration.

The lattice evaluation of the  $QQQ$  static potential has a long tradition (see e.g. [5] and references therein), while the static potential between a  $QQ$  pair in the presence of a light quark has been evaluated on the lattice only recently [6]. Expressions for the  $1/m$  and the  $1/m^2$  spin-dependent  $QQQ$  potentials in terms of Wilson loops can be found in [2], but have not been calculated on the lattice yet (while the complete expressions of all the  $Q\bar{Q}$  potentials up to order  $1/m^2$  can be found in [7] and the most recent lattice determinations are in [8]). Perturbative studies of the static potential may help to understand the transition region from the perturbative to the non-perturbative regime. In the case of the  $Q\bar{Q}$  potential, this region is characterized by the smooth transition from a Coulomb potential to a linear raising one (for recent studies, see [9]). In the case of the  $QQQ$  static potential, the transition from the perturbative to the non-perturbative regime is accompanied by the emergence of a three-body potential that depends on one length only (see, for instance, [5]). This is a rather spectacular phenomenon, which has been investigated only recently from a perturbative perspective [10].

There is so far no experimental evidence of  $QQQ$  baryons, while a few years ago the SELEX experiment has claimed evidence of possible doubly charmed baryon states [11]. Until today this evidence has not been confirmed by other experiments, but is mostly behind the revival of interest in this kind of systems at the mid of this decade.

## 2 EFT for $QQq$

The EFT Lagrangian that describes  $QQq$  baryons below the momentum transfer scale  $p$ , assumed to be larger than  $\Lambda_{\text{QCD}}$ , has the general form of Eq. (1), with  $\delta\mathcal{L}^{(i,j)}$  made out of four quark fields [2].

The term  $\delta\mathcal{L}^{(0,0)}$  is

$$\delta\mathcal{L}^{(0,0)} = \int d^3r T^\dagger \left[ iD_0 - V_T^{(0)} \right] T + \Sigma^\dagger \left[ iD_0 - V_\Sigma^{(0)} \right] \Sigma, \quad (2)$$

where  $T = (T^1, T^2, T^3)$  are the three independent  $QQ$  antitriplet fields,  $\Sigma = (\Sigma^1, \Sigma^2, \dots, \Sigma^6)$  are the six independent  $QQ$  sextet fields and the gauge fields in the covariant derivatives acting on the antitriplet and sextet fields are understood in the  $\bar{3}$  and 6 representations respectively. The matching coefficients  $V_T^{(0)}$  and  $V_\Sigma^{(0)}$  can be identified with the antitriplet and sextet static potentials respectively. The  $QQ$  antitriplet static potential  $V_T^{(0)}$  has been calculated recently up to next-to-next-to-leading order (NNLO) [10]; it reads

$$\begin{aligned} V_T^{(0)} = & -\frac{2}{3} \frac{\alpha_s(1/|\mathbf{r}|)}{|\mathbf{r}|} \left\{ 1 + \frac{\alpha_s(1/|\mathbf{r}|)}{4\pi} \left[ \frac{31}{3} + 22\gamma_E - \left( \frac{10}{9} + \frac{4}{3}\gamma_E \right) n_f \right] \right. \\ & + \left( \frac{\alpha_s}{4\pi} \right)^2 \left[ 66\zeta(3) + 484\gamma_E^2 + \frac{1976}{3}\gamma_E + \frac{3}{4}\pi^4 + \frac{121}{3}\pi^2 + \frac{4343}{18} \right. \\ & - \left( \frac{52}{3}\zeta(3) + \frac{176}{3}\gamma_E^2 + \frac{916}{9}\gamma_E + \frac{44}{9}\pi^2 + \frac{1229}{27} \right) n_f \\ & \left. \left. + \left( \frac{16}{9}\gamma_E^2 + \frac{80}{27}\gamma_E + \frac{4}{27}\pi^2 + \frac{100}{81} \right) n_f^2 \right] \right\}, \quad (3) \end{aligned}$$

where  $\alpha_s$  is the strong-coupling constant in the  $\overline{\text{MS}}$  scheme. The  $QQ$  sextet static potential  $V_\Sigma^{(0)}$  is repulsive at leading order:  $V_\Sigma^{(0)} = \alpha_s/(3|\mathbf{r}|)$ .

Several terms contribute to  $\delta\mathcal{L}^{(1,0)}$ . The one responsible for the hyperfine splitting is

$$\delta\mathcal{L}_{\text{hfs}}^{(1,0)} = \frac{V_{T\sigma\cdot BT}^{(1,0)}}{2} T^\dagger \frac{c_F}{2m} \left( \boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)} \right) \cdot g\mathbf{B}^a T_3^a T, \quad (4)$$

where  $V_{T\sigma\cdot BT}^{(1,0)} = 1 + \mathcal{O}(\alpha_s^2)$  is a matching coefficient of the EFT,  $c_F = 1 + \dots$  is the matching coefficient of the chromomagnetic interaction in the heavy quark effective theory, which is known up to three loops [12],  $\boldsymbol{\sigma}^{(i)}$  is a Pauli matrix acting on the heavy quark labeled  $i$  and  $T_3^a$  are the Gell-Mann matrices in the  $\bar{3}$  representation. From Eq. (4), it follows that the hyperfine splitting between the  $S$ -wave ground state of a doubly heavy baryon of spin 1/2 ( $\Xi_{QQ}$ ) and the corresponding state of spin 3/2 ( $\Xi_{QQ}^*$ ) may be related to the hyperfine splitting between the  $S$ -wave ground state of a heavy-light meson of spin 0 ( $P_{Q'}$ ) and the corresponding state of spin 1 ( $P_{Q'}^*$ ):

$$M_{\Xi_{QQ}^*} - M_{\Xi_{QQ}} = \frac{3m_{Q'}}{4m_Q} \frac{c_F^{(Q)}}{c_F^{(Q')}} \left( M_{P_{Q'}^*} - M_{P_{Q'}} \right) \left[ 1 + \mathcal{O} \left( \alpha_s^2, \frac{\Lambda_{\text{QCD}}}{m_Q}, \frac{\Lambda_{\text{QCD}}}{m_{Q'}} \right) \right], \quad (5)$$

where we have kept different the mass,  $m_Q$ , of the heavy quarks in the baryons from the one,  $m_{Q'}$ , in the mesons. The obtained figures compare well with existing lattice determinations (see discussion and references in [2], a more recent unquenched lattice determination of  $M_{\Xi_{bb}^*} - M_{\Xi_{bb}}$ , which is consistent with previous quenched determinations, may be found in [13]).

Other terms in the effective Lagrangian have been derived in [2].

### 3 EFT for $QQQ$

The EFT Lagrangian that describes  $QQQ$  baryons below the momentum transfer scale  $p$ , assumed to be larger than  $\Lambda_{\text{QCD}}$ , has the general form of Eq. (1), with  $\delta\mathcal{L}^{(i,j)}$  made out of six quark fields [2].

The term  $\delta\mathcal{L}^{(0,0)}$  is

$$\delta\mathcal{L}^{(0,0)} = \int d^3r_1 d^3r_2 S^\dagger \left[ i\partial_0 - V_S^{(0)} \right] S + O^\dagger \left[ iD_0 - V_O^{(0)} \right] O + \Delta^\dagger \left[ iD_0 - V_\Delta^{(0)} \right] \Delta, \quad (6)$$

where  $S$  is the singlet field,  $O = \begin{pmatrix} O^A \\ O^S \end{pmatrix}$ , with  $O^A = (O^{A1}, O^{A2}, \dots, O^{A8})$  and  $O^S = (O^{S1}, O^{S2}, \dots, O^{S8})$ , are the fields that parameterize the two possible octet configurations of three quarks,  $\Delta = (\Delta^1, \Delta^2, \dots, \Delta^{10})$  are the ten independent  $QQQ$  decuplet fields and the gauge fields in the covariant derivatives acting on the octets and decuplet fields are understood in the 8 and 10 representations respectively. The matching coefficients  $V_S^{(0)}$ ,  $V_O^{(0)}$  and  $V_\Delta^{(0)}$  can be identified with the singlet, octet and decuplet static potentials respectively; note that the octet potential  $V_O^{(0)}$  is a  $2 \times 2$  matrix. The  $QQQ$  singlet static potential  $V_S^{(0)}$  has been calculated up to NNLO in [10]; it reads

$$\begin{aligned} V_S^{(0)} = & -\frac{2}{3} \sum_{q=1}^3 \frac{\alpha_s(1/|\mathbf{r}_q|)}{|\mathbf{r}_q|} \left\{ 1 + \frac{\alpha_s(1/|\mathbf{r}_q|)}{4\pi} \left[ \frac{31}{3} + 22\gamma_E - \left( \frac{10}{9} + \frac{4}{3}\gamma_E \right) n_f \right] \right. \\ & + \left( \frac{\alpha_s}{4\pi} \right)^2 \left[ 66\zeta(3) + 484\gamma_E^2 + \frac{1976}{3}\gamma_E + \frac{3}{4}\pi^4 + \frac{121}{3}\pi^2 + \frac{4343}{18} \right. \\ & \quad \left. - \left( \frac{52}{3}\zeta(3) + \frac{176}{3}\gamma_E^2 + \frac{916}{9}\gamma_E + \frac{44}{9}\pi^2 + \frac{1229}{27} \right) n_f \right. \\ & \quad \left. \left. + \left( \frac{16}{9}\gamma_E^2 + \frac{80}{27}\gamma_E + \frac{4}{27}\pi^2 + \frac{100}{81} \right) n_f^2 \right] \right\} \\ & + V_S^{\text{3body}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \end{aligned} \quad (7)$$

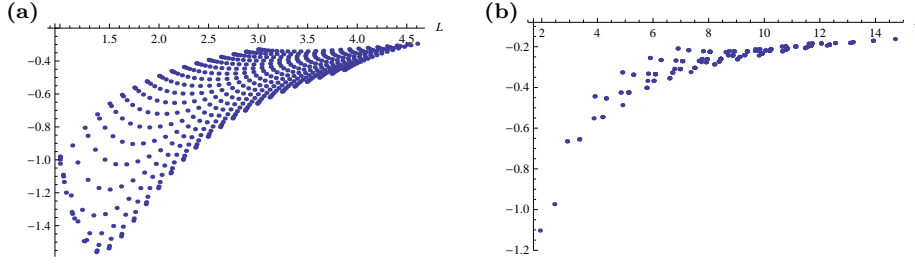
where  $\mathbf{r}_1$ ,  $\mathbf{r}_2$  and  $\mathbf{r}_3$  are the distances between the heavy quarks; only two of them are independent: if we call  $\mathbf{x}_1$ ,  $\mathbf{x}_2$  and  $\mathbf{x}_3$  the coordinates of the three quarks, our choice is  $\mathbf{r}_1 = \mathbf{x}_1 - \mathbf{x}_2$ ,  $\mathbf{r}_2 = \mathbf{x}_1 - \mathbf{x}_3$  and  $\mathbf{r}_3 = \mathbf{x}_2 - \mathbf{x}_3$ , which implies that  $\mathbf{r}_3 = \mathbf{r}_2 - \mathbf{r}_1$ .  $V_S^{\text{3body}}$  is the three-body part of the perturbative potential, defined as the part of the potential that vanishes when putting one of the quarks at infinite distance from the other two.

The three-body part is a specific feature of the two-loop potential; it reads

$$V_S^{\text{3body}} = -\alpha_s \left( \frac{\alpha_s}{4\pi} \right)^2 [v(\mathbf{r}_2, \mathbf{r}_3) + v(\mathbf{r}_1, -\mathbf{r}_3) + v(-\mathbf{r}_2, -\mathbf{r}_1)], \quad (8)$$

where

$$\begin{aligned} v(\boldsymbol{\rho}, \boldsymbol{\lambda}) = & 16\pi \hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\lambda}} \int_0^1 dx \int_0^1 dy \frac{1}{R} \left[ \left( 1 - \frac{M^2}{R^2} \right) \arctan \frac{R}{M} + \frac{M}{R} \right] \\ & + 16\pi \hat{\boldsymbol{\rho}}^i \hat{\boldsymbol{\lambda}}^j \int_0^1 dx \int_0^1 dy \frac{\hat{\mathbf{R}}^i \hat{\mathbf{R}}^j}{R} \left[ \left( 1 + 3 \frac{M^2}{R^2} \right) \arctan \frac{R}{M} - 3 \frac{M}{R} \right], \end{aligned} \quad (9)$$



**Fig. 1** The normalized three-body potential,  $2V_S^{3\text{body}}/\alpha_s^3$ , plotted as function of  $L$  in arbitrary units for the two geometries described in the text.

with  $\mathbf{R} = x\boldsymbol{\rho} - y\boldsymbol{\lambda}$ ,  $R = |\mathbf{R}|$  and  $M = |\boldsymbol{\rho}|\sqrt{x(1-x)} + |\boldsymbol{\lambda}|\sqrt{y(1-y)}$ . This term has a different dependence on the positions of the three quarks with respect to the Coulomb potential. It is finite for all configurations: it vanishes when one of the quarks is pulled at infinite distance (note that in this case  $V_S^{(0)}$  becomes  $V_T^{(0)}$ ) and still remains finite when two quarks are put in the same position (note that in this case  $V_S^{(0)}$  becomes the  $Q\bar{Q}$  singlet static potential). Hence, its dependence on the geometry is much smoother than the dependence on the geometry of the Coulomb potential, although it still clearly depends on it. These features can be seen by plotting  $V_S^{3\text{body}}$  as a function of  $L$  for two different set of configurations;  $L$  is the sum of the distances of the three quarks from the so-called Fermat (or Torricelli) point, which has minimum distance from the quarks. Figure 1(a) shows the three-body potential when we place the three quarks in a plane  $(x, y)$ , fix the position of the first quark in  $(0, 0)$ , the position of the second one in  $(1, 0)$  and move the third one in  $(0.5 + 0.125n_x, 0.125n_y)$  for  $n_x \in \{0, 1, \dots, 20\}$  and  $n_y \in \{0, 1, \dots, 24\}$ . The plot shows a clear dependence on the geometry at fixed  $L$  (i.e. different configurations with same  $L$  give different potentials), although weaker than in the Coulombic case. Figure 1(b) shows the three-body potential in a geometry used in [5], which consists in placing the three quarks along the axes of a three-dimensional lattice:  $(n_x, 0, 0)$ ,  $(0, n_y, 0)$  and  $(0, 0, n_z)$ , for  $n_x \in \{0, 1, \dots, 6\}$  and  $n_y, n_z \in \{1, \dots, 6\}$ . The plot shows a weaker dependence on the geometry than in the previous case. As the comparison with Fig. 1(a) indicates, the weaker dependence is an artifact of the chosen configurations rather than a physical effect. Clearly, the precise identification of the transition region from a two-body dominated potential to a three-body dominated one would require to perform lattice calculations in geometries different from the one used in Fig. 1(b). The use of different geometries could also be important to assess the nature of the long-range three-body potential.

The one-gluon exchange mixes the octet fields, so that the octet potential  $V_O^{(0)}$  is a non-diagonal  $2 \times 2$  matrix already at leading order. Choosing  $O^S$  and  $O^A$  to be respectively symmetric and antisymmetric for exchanges of the quarks located in  $\mathbf{x}_1$  and  $\mathbf{x}_2$  (a different choice of the octet fields would correspond to a field redefinition leading to a different octet potential), we obtain

$$V_O^{(0)} = \alpha_s \left[ \frac{1}{|\mathbf{r}_1|} \begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} + \frac{1}{|\mathbf{r}_2|} \begin{pmatrix} \frac{1}{12} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{5}{12} \end{pmatrix} + \frac{1}{|\mathbf{r}_3|} \begin{pmatrix} \frac{1}{12} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{5}{12} \end{pmatrix} \right], \quad (10)$$

while at leading order the decuplet potential is

$$V_{\Delta}^{(0)} = \frac{\alpha_s}{3} \left( \frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|} + \frac{1}{|\mathbf{r}_3|} \right). \quad (11)$$

We mention that there exist lattice data that show very clearly the singlet, octet and decuplet  $QQQ$  static potentials, although in an equilateral geometry, where the two octets are degenerate [14].

Other terms in the effective Lagrangian have been derived in [2].

## 4 Conclusions

Systems made of two or three heavy quarks or antiquarks develop similar hierarchies of energy scales and may be treated in similar EFT frameworks.

In the case of  $Q\bar{Q}$  mesons, the static potential has been determined up to next-to-next-to-next-to-leading order in perturbation theory and to a high accuracy on the lattice. The Coulomb behaviour starts getting substantial modifications at distances around 0.2 fm turning over a linearly raising potential at larger distances. Therefore the transition region can be studied to a large extent with perturbative methods. Terms proportional to powers of  $1/m$  and  $r$  have been also calculated and included systematically in physical observables.

In the case of  $QQq$  baryons, the static potential has been determined up to NNLO in perturbation theory and recently also on the lattice. Terms proportional to powers of  $1/m$  and  $r$  in the Lagrangian have been matched (mostly) at leading order and used to determine, for instance, the expected hyperfine splitting of the ground state of these systems. If early experimental evidences will get confirmation in future experimental facilities, this will mark the beginning of a future new spectroscopy.

Finally, in the case of  $QQQ$  baryons, the static potential has been determined up to NNLO in perturbation theory and also on the lattice. The transition region from a Coulomb to a linearly raising potential is characterized in this case also by the emergence of a three-body potential apparently parameterized by only one length. While we have argued that more lattice studies employing different geometries would be necessary to precisely identify the transition region, we have also shown that in perturbation theory a smooth genuine three-body potential shows up at two loops.

**Acknowledgements** I acknowledge financial support from the RTN Flavianet MRTN-CT-2006-035482 (EU) and from the DFG cluster of excellence “Origin and structure of the universe” (<http://www.universe-cluster.de>).

## References

1. N. Brambilla, A. Pineda, J. Soto and A. Vairo, “Effective field theories for heavy quarkonium,” *Rev. Mod. Phys.* **77**, 1423 (2005) [arXiv:hep-ph/0410047].
2. N. Brambilla, T. Röscher and A. Vairo, “Effective field theory Lagrangians for baryons with two and three heavy quarks”, *Phys. Rev. D* **72**, 034021 (2005) [arXiv:hep-ph/0506065].
3. S. Fleming and T. Mehen, “Doubly Heavy Baryons, Heavy Quark-DiQuark Symmetry and NRQCD,” *Phys. Rev. D* **73**, 034502 (2006) [arXiv:hep-ph/0509313].
4. M. J. Savage and M. B. Wise, “Spectrum Of Baryons With Two Heavy Quarks,” *Phys. Lett. B* **248**, 177 (1990).

5. T. T. Takahashi, H. Matsufuru, Y. Nemoto and H. Suganuma, “The three-quark potential in the SU(3) lattice QCD,” *Phys. Rev. Lett.* **86**, 18 (2001) [hep-lat/0006005]; “Detailed analysis of the three quark potential in SU(3),” *Phys. Rev. D* **65**, 114509 (2002) [hep-lat/0204011];
6. A. Yamamoto, H. Suganuma and H. Iida, “Heavy-heavy-light quark potential in SU(3) lattice QCD,” *Phys. Lett. B* **664**, 129 (2008) [arXiv:0708.3610 [hep-lat]]; “Lattice QCD study of the heavy-heavy-light quark potential,” *Phys. Rev. D* **78**, 014513 (2008) [arXiv:0806.3554 [hep-lat]]; J. Najjar and G. Bali, “Static-static-light baryonic potentials,” arXiv:0910.2824 [hep-lat].
7. N. Brambilla, A. Pineda, J. Soto and A. Vairo, “The QCD potential at  $O(1/m)$ ,” *Phys. Rev. D* **63**, 014023 (2001) [arXiv:hep-ph/0002250]; A. Pineda and A. Vairo, “The QCD potential at  $O(1/m^2)$  : Complete spin dependent and spin independent result,” *Phys. Rev. D* **63**, 054007 (2001) [Erratum-ibid. *D* **64**, 039902 (2001)] [arXiv:hep-ph/0009145]; N. Brambilla, A. Pineda, J. Soto and A. Vairo, “The  $(m\Lambda_{\text{QCD}})^{1/2}$  scale in heavy quarkonium,” *Phys. Lett. B* **580**, 60 (2004) [arXiv:hep-ph/0307159].
8. Y. Koma, M. Koma and H. Wittig, “Nonperturbative determination of the QCD potential at  $O(1/m)$ ,” *Phys. Rev. Lett.* **97**, 122003 (2006) [arXiv:hep-lat/0607009]; Y. Koma and M. Koma, “Spin-dependent potentials from lattice QCD,” *Nucl. Phys. B* **769**, 79 (2007) [arXiv:hep-lat/0609078]; Y. Koma, M. Koma and H. Wittig, “Relativistic corrections to the static potential at  $O(1/m)$  and  $O(1/m^2)$ ,” *PoS LAT2007*, 111 (2007) [arXiv:0711.2322 [hep-lat]]; Y. Koma and M. Koma, “Scaling study of the relativistic corrections to the static potential,” arXiv:0911.3204 [hep-lat].
9. N. Brambilla, X. Garcia i Tormo, J. Soto and A. Vairo, “Precision determination of  $r_0\Lambda_{\overline{\text{MS}}}$  from the QCD static energy,” arXiv:1006.2066 [hep-ph]; “The QCD static energy at NNLL,” *Phys. Rev. D* **80**, 034016 (2009) [arXiv:0906.1390 [hep-ph]].
10. N. Brambilla, J. Ghiglieri and A. Vairo, “The three-quark static potential in perturbation theory,” *Phys. Rev. D* **81**, 054031 (2010) [arXiv:0911.3541 [hep-ph]].
11. M. Mattson *et al.* [SELEX Collaboration], “First observation of the doubly charmed baryon  $\Xi_{cc}^+$ ,” *Phys. Rev. Lett.* **89**, 112001 (2002) [arXiv:hep-ex/0208014]; A. Ocherashvili *et al.* [SELEX Collaboration], “Confirmation of the double charm baryon  $\Xi_{cc}^+(3520)$  via its decay to  $pD^+K^-$ ,” *Phys. Lett. B* **628**, 18 (2005) [arXiv:hep-ex/0406033].
12. A. G. Grozin, P. Marquard, J. H. Piclum and M. Steinhauser, “Three-Loop Chromomagnetic Interaction in HQET,” *Nucl. Phys. B* **789**, 277 (2008) [arXiv:0707.1388 [hep-ph]].
13. R. Lewis and R. M. Woloshyn, “Bottom baryons from a dynamical lattice QCD simulation,” *Phys. Rev. D* **79**, 014502 (2009) [arXiv:0806.4783 [hep-lat]].
14. K. Hübner, F. Karsch, O. Kaczmarek and O. Vogt, “Heavy quark free energies for three quark systems at finite temperature,” *Phys. Rev. D* **77**, 074504 (2008) [arXiv:0710.5147 [hep-lat]].